

POSTNIKOV PIECES AND $B\mathbb{Z}/p$ -HOMOTOPY THEORY

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ABSTRACT. We present a constructive method to compute the cellularization with respect to $B^m\mathbb{Z}/p$ for any integer $m \geq 1$ of a large class of H -spaces, namely all those which have a finite number of non-trivial $B^m\mathbb{Z}/p$ -homotopy groups (the pointed mapping space $\mathrm{map}_*(B^m\mathbb{Z}/p, X)$ is a Postnikov piece). We prove in particular that the $B^m\mathbb{Z}/p$ -cellularization of an H -space having a finite number of $B^m\mathbb{Z}/p$ -homotopy groups is a p -torsion Postnikov piece. Along the way we characterize the $B\mathbb{Z}/p^r$ -cellular classifying spaces of nilpotent groups.

INTRODUCTION

The notion of A -homotopy theory was introduced by Dror Farjoun [9] for an arbitrary connected space A . Here A and its suspensions play the role of the spheres in classical homotopy theory and so the A -homotopy groups of a space X are defined to be the homotopy classes of pointed maps $[\Sigma^i A, X]$. The analogue to weakly contractible spaces are those spaces for which all A -homotopy groups are trivial. This means that the pointed mapping space $\mathrm{map}_*(A, X)$ is contractible, i.e. X is an A -local space. On the other hand the classical notion of CW -complex is replaced by the one of A -cellular space. Such spaces that can be constructed from A by means of pointed homotopy colimits.

Thanks to work of Bousfield [2] and Dror Farjoun [9] there is a functorial way to study X through the eyes of A . The nullification $P_A X$ is the biggest quotient of X which is A -local and $CW_A X$ is the best A -cellular approximation of the space X . Roughly speaking $CW_A X$ contains all the transcendent information of the mapping space $\mathrm{map}_*(A, X)$ since it is equivalent to $\mathrm{map}_*(A, CW_A X)$. Hence explicit computation of the cellularization would give access to information about $\mathrm{map}_*(A, X)$. The importance of mapping spaces (in the case $A = B\mathbb{Z}/p$) is well established since Miller's solution to the Sullivan conjecture [17].

While there is a lot of literature devoted to computations of $P_A X$, only very few computations of $CW_A X$ are available. For instance Chachólski describes a strategy to compute the cellularization $CW_A(X)$ in [7]. This method has been successfully applied in some cases (cellularization with respect to Moore spaces [21], $B\mathbb{Z}/p$ -cellularization of classifying spaces of finite groups [10]), but it is in general difficult to apply.

An alternative way to compute $CW_A X$ is the following. The localization map $l : X \rightarrow P_A X$ provides an equivalence $CW_A X \simeq CW_A \overline{P}_A X$ where as usual $\overline{P}_A X$ denotes

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the homotopy fiber of l . This equivalence gives a strategy to compute cellularizations when $CW_A \overline{P}_A X$ is known. For instance if X is A -local then $\overline{P}_A X \simeq *$ and so $CW_A X \simeq *$. From the A -homotopy point of view the next case in which the A -cellularization should be accessible is when X has only a finite number of A -homotopy groups, that is some iterated loop space $\Omega^n X$ is A -local. Natural examples of spaces satisfying this condition are obtained by considering the n -connected covers of A -local spaces.

Let us specialize in H -spaces and $A = B^m \mathbb{Z}/p$. P. Bousfield has determined the fiber of the localization map $X \rightarrow P_{B^m \mathbb{Z}/p} X$ (see [2]) when $\Omega^n X$ is $B^m \mathbb{Z}/p$ -local. He shows that for such an H -space $\overline{P}_{B^m \mathbb{Z}/p} X$ is a p -torsion Postnikov piece F , whose homotopy groups are concentrated in degrees from m to $m+n-1$. As F is also an H -space (because l is an H -map), we call it an *H -Postnikov piece*. The cellularization of X (which is again an H -space because CW_A preserves H -structures) therefore coincides with that of a Postnikov piece. We do this in Section 3 and this enables us to obtain our main result.

Theorem 4.1 *Let X be a connected H -space such that $\Omega^n X$ is $B^m \mathbb{Z}/p$ -local. Then*

$$CW_{B^m \mathbb{Z}/p} X \simeq F \times K(W, m)$$

where F is a p -torsion H -Postnikov piece with homotopy concentrated in degrees from $m+1$ to n and W is an elementary abelian p -group.

Thus, when X is an H -space with only a finite number of $B^m \mathbb{Z}/p$ -homotopy groups, the cellularization $CW_{B^m \mathbb{Z}/p} X$ is a p -torsion H -Postnikov piece! This is not true in general if we do not assume X to be an H -space. For instance, the $B\mathbb{Z}/p$ -cellularization of $B\Sigma_3$ is a space with infinitely many non-trivial homotopy groups [11].

For $m = 1$ there is a large class of H -spaces which is known to have some local loop space by previous work of the authors [6]: those for which the mod p cohomology is finitely generated as an algebra over the Steenrod algebra. Hence we obtain the following.

Proposition 4.2 *Let X be a connected H -space such that $H^*(X; \mathbb{F}_p)$ is finitely generated as algebra over the Steenrod algebra. Then*

$$CW_{B\mathbb{Z}/p} X \simeq F \times K(W, 1)$$

where F is a 1-connected p -torsion H -Postnikov piece and W is an elementary abelian p -group. Moreover, there exists an integer k such that $CW_{B^m \mathbb{Z}/p} X \simeq *$ for $m \geq k$.

Our results allow explicit computations which we exemplify by computing the $B\mathbb{Z}/p$ -cellularization of the n -connected cover of any finite H -spaces (Proposition 4.3), as well as the $B^m \mathbb{Z}/p$ -cellularizations of the classifying spaces for real and complex vector bundles BU , BO , and their connected covers BSU , BSO , $BSpin$, and $BString$, see Proposition 5.6.

1. A DOUBLE FILTRATION OF THE CATEGORY OF SPACES

As mentioned in the introduction the condition that $\Omega^n X$ be $B^m\mathbb{Z}/p$ -local will enable us to compute the $B^m\mathbb{Z}/p$ -cellularization of H -spaces. This section is devoted to give a picture of how such spaces are related for different choices of m and n .

First of all we present a technical lemma which collects various facts needed in the rest of the paper.

Lemma 1.1. *Let X be a connected space and $m > 0$. Then,*

- (1) *If X is $B^m\mathbb{Z}/p$ -local then $\Omega^n X$ is $B^m\mathbb{Z}/p$ -local for all $n \geq 1$.*
- (2) *If X is $B^m\mathbb{Z}/p$ -local then it is $B^{m+s}\mathbb{Z}/p$ -local for all $s \geq 0$.*
- (3) *If ΩX is $B^m\mathbb{Z}/p$ -local, then X is $B^{m+s}\mathbb{Z}/p$ -local for all $s \geq 1$.*

Proof. To prove (1) simply apply $\text{map}_*(B\mathbb{Z}/p, -)$ to the path fibration $\Omega X \rightarrow * \rightarrow X$.

Statement (2) is given by Dwyer's version of Zabrodsky's lemma [8, Prop. 3.4] to the universal fibration $B^n\mathbb{Z}/p \rightarrow * \rightarrow B^{n+1}\mathbb{Z}/p$.

Finally (3) is a direct consequence of Zabrodsky's lemma (now in its connected version [8, Prop. 3.5]) applied to the universal fibration and using the fact that ΩX $B^n\mathbb{Z}/p$ -local implies $\text{map}(B^n\mathbb{Z}/p, X)_c \simeq X$. ■

Of course the converses of the previous results are not true. For the first statement take the classifying space of a discrete group at $m = 1$. For the second and third consider $X = BU$. It is a $B^2\mathbb{Z}/p$ -local (see Example 1.4) space but neither BU nor ΩBU are $B\mathbb{Z}/p$ -local. Observe that in fact $\Omega^n BU$ is never $B\mathbb{Z}/p$ -local. The next result shows that this is the general situation for H -spaces. That is, if an H -space is $B^{m+1}\mathbb{Z}/p$ -local then either ΩX is $B^m\mathbb{Z}/p$ -local or $\Omega^n X$ is never $B^m\mathbb{Z}/p$ -local $\forall n \geq 1$.

Theorem 1.2. *Let X be a $B^{n+1}\mathbb{Z}/p$ -local space such that $\Omega^k X$ is $B^n\mathbb{Z}/p$ -local for some $k > 0$. Then ΩX is $B^n\mathbb{Z}/p$ -local.*

Proof. It is enough to prove the result for $k = 2$. Consider the fibration

$$K(Q, n+1) \longrightarrow P_{\Sigma^2 B^n\mathbb{Z}/p} X \simeq X \longrightarrow P_{\Sigma B^n\mathbb{Z}/p} X$$

where the fiber is a p -torsion Eilenberg-Mac Lane space by Bousfield's description of the fiber of the $\Sigma B^n\mathbb{Z}/p$ -nullification [2, Theorem 7.2]. The total space is $B^{n+1}\mathbb{Z}/p$ -local and so is the base by the previous lemma. Thus $\text{map}_*(B^{n+1}\mathbb{Z}/p, K(Q, n+1))$ must be contractible as well, i.e. $Q = 0$. ■

The previous analysis leads to a double filtration of the category of spaces. Let $n \geq 0$ and $m \geq 1$. We introduce the notation

$$\mathcal{S}_m^n = \{X; \Omega^n X \text{ is } B^m\mathbb{Z}/p\text{-local}\}.$$

Lemma 1.1 yields then a diagram of inclusions:

$$\begin{array}{ccccccc}
\mathcal{S}_1^0 & \hookrightarrow & \mathcal{S}_1^1 & \hookrightarrow & \mathcal{S}_1^2 & \hookrightarrow & \dots \hookrightarrow \mathcal{S}_1^n \dots \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \nearrow \\
\mathcal{S}_2^0 & \hookrightarrow & \mathcal{S}_2^1 & \hookrightarrow & \mathcal{S}_2^2 & \hookrightarrow & \dots \hookrightarrow \mathcal{S}_2^n \dots \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \nearrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\mathcal{S}_m^0 & \hookrightarrow & \mathcal{S}_m^1 & \hookrightarrow & \mathcal{S}_m^2 & \hookrightarrow & \dots \hookrightarrow \mathcal{S}_m^n \dots \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Example 1.3. Examples of spaces in every stage of the filtration are known.

- (1) \mathcal{S}_1^0 are the spaces that are $B\mathbb{Z}/p$ -local. This contains in particular any finite space (by Miller's theorem [17, Thm. A]), and for a nilpotent space X (of finite type with finite fundamental group) to be $B\mathbb{Z}/p$ -local is equivalent to its cohomology $H^*(X; \mathbb{F}_p)$ being locally finite by [22, Corollary 8.6.2].
- (2) If $X\langle n \rangle$ denotes the n -connected cover of a space X , then the homotopy fiber of $\Omega^{n-1}X\langle n \rangle \rightarrow \Omega^{n-1}X$ is a discrete space. Hence if $X \in \mathcal{S}_m^0$ then $X\langle n \rangle \in \mathcal{S}_m^{n-1}$.
- (3) Observe that $\mathcal{S}_m^n \subset \mathcal{S}_{m+k}^{n-k}$ for all $0 \leq k \leq n$.
- (4) The previous examples provide spaces in every stage of the double filtration. Consider a finite space. It is automatically $B\mathbb{Z}/p$ -local. Its n -connected cover $X\langle n \rangle$ lies in \mathcal{S}_1^{n-1} . Hence $X \in \mathcal{S}_{k+1}^{n-k-1}$ for all $0 \leq k \leq n$.

Next example provides a number of spaces living in \mathcal{S}_m^0 but not obtained from the first row of the filtration by taking n -connected covers. Of course their connected covers will be *new* examples of spaces living in \mathcal{S}_m^n .

Example 1.4. Let E_* be a homology theory. If $\tilde{E}^i(K(\mathbb{Z}/p\mathbb{Z}, n)) = 0$ for $i \geq j$ then the spaces E^i for $i \geq j$ representing the corresponding homology theory are $B^n\mathbb{Z}/p$ -local. If $\tilde{E}^j(K(\mathbb{Z}/p\mathbb{Z}, n-1)) \neq 0$ then E^j is not $B^{n-1}\mathbb{Z}/p$ -local. In particular if E_* is periodic, it follows that the spaces $\{E^i\}$ for $i \geq j$ are $B^n\mathbb{Z}/p$ -local but none of their iterated loops are $B^{n-1}\mathbb{Z}/p$ -local.

A first example of such a behavior is obtained from complex K-theory, BU is $B^2\mathbb{Z}/p$ -local but BU and U are not $B\mathbb{Z}/p$ -local (see [18]). Note that real and quaternionic K-theory enjoy the same properties.

For every n , examples of homology theories following this pattern are given by p -torsion homology theories of type III- n as described in [1]. The n th Morava K-theory $K(n)_*$ for p odd is an example of such behavior with respect to Eilenberg-Mac Lane spaces. The spaces representing $K(n)_*$ are $B^{n+1}\mathbb{Z}/p$ -local but none of their iterated loops are $B^n\mathbb{Z}/p$ -local.

Our aim is to provide tools to compute the $B^m\mathbb{Z}/p$ -cellularization of any H -space lying in the m -th row of the above diagram. The key point is the following result of Bousfield [2] determining the fiber of the localization map.

Proposition 1.5. *Let $n \geq 0$ and X be a connected H -space X such that $\Omega^n X$ is $B^m\mathbb{Z}/p$ -local. Then there is an H -fibration*

$$F \longrightarrow X \longrightarrow P_{B^m\mathbb{Z}/p}X$$

where F is a p -torsion H -Postnikov piece whose homotopy groups are concentrated in degrees from m to $m+n$. ■

Therefore, since $F \rightarrow X$ is a $B^m\mathbb{Z}/p$ -cellular equivalence, we only need to compute the cellularization of a Postnikov piece (which will end up being a Postnikov piece again, see Theorem 3.6). Actually even more is true.

Proposition 1.6. *Let X be a connected space such that $CW_{B^m\mathbb{Z}/p}X$ is a Postnikov piece. Then there exists an integer n such that $\Omega^n X$ is $B^m\mathbb{Z}/p$ -local.*

Proof. Let us loop once Chachólski fibration $CW_{B^m\mathbb{Z}/p}X \rightarrow X \rightarrow P_{\Sigma B^m\mathbb{Z}/p}C$, see [7, Theorem 20.5]. As $\Omega P_{\Sigma B^m\mathbb{Z}/p}C$ is equivalent to $P_{B^m\mathbb{Z}/p}\Omega C$ by [9, Theorem 3.A.1], we get a fibration over a $B^m\mathbb{Z}/p$ -local base space

$$\Omega CW_{B^m\mathbb{Z}/p}X \longrightarrow \Omega X \longrightarrow P_{B^m\mathbb{Z}/p}\Omega C.$$

Now there exists an integer n such that $\Omega^n CW_{B^m\mathbb{Z}/p}X$ is discrete, thus $B^m\mathbb{Z}/p$ -local. Therefore so is $\Omega^n X$. ■

2. CELLULARIZATION OF FIBRATIONS OVER BG

In general the cellularization of the total space of a fibration is very difficult to compute. We explain in this section how to deal with this problem when the base space is the classifying space of a discrete group. The first step applies to any group, in the second, see Proposition 2.4 below, we specialize to nilpotent groups.

Proposition 2.1. *Let $r \geq 1$ and $F \longrightarrow E \xrightarrow{\pi} BG$ be a fibration where G is a discrete group. Let S be the (normal) subgroup generated by all elements $g \in G$ of order p^i for some $i \leq r$ such that the inclusion $B\langle g \rangle \longrightarrow BG$ lifts to E up to an unpointed homotopy. Then the pullback of the fibration along $BS \longrightarrow BG$*

$$\begin{array}{ccccc} E' & \xrightarrow{f} & E & \xrightarrow{p} & B(G/S) \\ \downarrow & & \downarrow \pi & & \downarrow \\ BS & \longrightarrow & BG & \xrightarrow{p'} & B(G/S) \end{array}$$

induces a $B\mathbb{Z}/p^r$ -cellular equivalence $f : E' \rightarrow E$ on the total space level.

Proof. We have to show that f induces a homotopy equivalence on pointed mapping spaces $\text{map}_*(B\mathbb{Z}/p^r, -)$. The top fibration in the diagram yields a fibration

$$\text{map}_*(B\mathbb{Z}/p^r, E') \xrightarrow{f_*} \text{map}_*(B\mathbb{Z}/p^r, E) \xrightarrow{p_*} \text{map}_*(B\mathbb{Z}/p^r, B(G/S)).$$

Since the base is homotopically discrete we only need to check that all components of the total space are sent by p_* to the component of the constant. Consider thus a map $h : B\mathbb{Z}/p^r \rightarrow E$. The composite $p \circ h$ is homotopy equivalent to a map induced by a group morphism $\alpha : \mathbb{Z}/p^r \longrightarrow G$ whose image $\alpha(1) = g$ is in S by construction. Therefore $p \circ h = p' \circ \pi \circ h$ is null-homotopic. \blacksquare

Remark 2.2. If the fibration in the above proposition is an H -fibration (in particular G is abelian), the set of elements g for which there is a lift to the total space forms a subgroup of G . The central extension $Z(D_8) \hookrightarrow D_8 \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$ of the dihedral group D_8 provides an example where the subgroup S is $\mathbb{Z}/2 \times \mathbb{Z}/2$ but the element in S represented by an element of order 4 in D_8 does not admit a lift.

The next lemma is a variation of Dwyer's version of Zabrodsky's Lemma in [8].

Lemma 2.3. *Let $F \rightarrow E \xrightarrow{f} B$ be a fibration over a connected base, and A a connected space such that ΩA is F -local. Then any map $g : E \rightarrow A$ which is homotopic to the constant when restricted to F factors through a map $h : B \rightarrow A$ up to unpointed homotopy and moreover g is pointed null-homotopic if and only if h is so.*

Proof. Since ΩA is F -local, we see that the component of the constant map $\text{map}_*(F, A)_c$ is contractible and therefore the evaluation at the base point $\text{map}(F, A)_c \rightarrow A$ is an equivalence. By Proposition 3.5 in [8], f induces a homotopy equivalence

$$\text{map}(B, A) \simeq \text{map}(E, A)_{[F]}.$$

where $\text{map}(E, A)_{[F]}$ denotes the space of those maps $E \rightarrow A$ which are homotopic to the constant when restricted to F .

We restrict now to the component of the constant map $c : E \rightarrow A$. There is only one component in the pointed mapping space sitting over c since any map homotopic to the constant map is also homotopic by a pointed homotopy. The result follows. \blacksquare

Proposition 2.4. *Let $r \geq 1$ and $F \xrightarrow{i} E \xrightarrow{\pi} BG$ be a fibration where G is a nilpotent group generated by elements of order p^i with $i \leq r$. Assume that for each of these generators $x \in G$, the inclusion $B\langle x \rangle \longrightarrow BG$ lifts to E up to unpointed homotopy. If F is $B\mathbb{Z}/p^r$ -cellular then so is E .*

Proof. Chachólski's description [7] of the cellularization $CW_{B\mathbb{Z}/p^r}(E)$ as the homotopy fiber of the composite $f : E \longrightarrow C \longrightarrow P_{\Sigma B\mathbb{Z}/p^r}(C)$ where C is the homotopy cofiber of the evaluation map $\bigvee_{[B\mathbb{Z}/p^r, E]} B\mathbb{Z}/p^r \longrightarrow E$ tells us that E is cellular if the map f is null-homotopic. Observe that if f is nullhomotopic then the fiber inclusion $CW_{B\mathbb{Z}/p^r}(E) \longrightarrow E$ has a section and therefore E is cellular since it is a retract of a cellular space ([9, 2.D.1.5]).

As the existence of an unpointed homotopy to the constant map implies the existence of a pointed one, we work now in the category of unpointed spaces. Remark that for any map $g : Z \longrightarrow E$ from a $B\mathbb{Z}/p^r$ -cellular space Z , the composite $f \circ g$ is null-homotopic since g factors through the cellularization of E . In particular the composite $f \circ i$ is null-homotopic. By Lemma 2.3 there exists $\bar{f} : BG \longrightarrow P_{\Sigma B\mathbb{Z}/p^r}(C)$ such that $\bar{f} \circ \pi \simeq f$ and, moreover, f is null-homotopic if and only if \bar{f} is so.

We first assume that G is a finite group and show by induction on the order of G that \bar{f} is null-homotopic. If $|G| = p$, then the existence of a section $s : BG \rightarrow E$ implies that $f \circ s = \bar{f}$ is nullhomotopic since $BG = B\mathbb{Z}/p$ is cellular.

Let $\{x_1, \dots, x_k\}$ be a minimal set of generators which admit a lift. Let $H \trianglelefteq G$ be the normal subgroup generated by x_1, \dots, x_{k-1} and their conjugates by powers of x_k . There is a short exact sequence $H \rightarrow G \rightarrow \mathbb{Z}/p^a$ where the quotient group is generated by the image of x_k . Consider the fibration $F \rightarrow E' \rightarrow BH$ obtained by pulling back along $BH \rightarrow BG$ and denote by $h : E' \rightarrow E$ the induced map between the total spaces. The inclusions in G of two conjugate subgroups are (freely) homotopic and so H satisfies the assumptions of the proposition. Thus the induction hypothesis tells us that E' is cellular and therefore $f \circ h$ is nullhomotopic. This implies that the restriction of \bar{f} to BH is nullhomotopic. Consider the following diagram

$$\begin{array}{ccccc}
 B(\langle x_k \rangle \cap H) & \longrightarrow & BH & & \\
 \downarrow & & \downarrow & \searrow * & \\
 B(\langle x_k \rangle) & \longrightarrow & BG & \xrightarrow{\bar{f}} & P_{\Sigma B\mathbb{Z}/p^r}(C) \\
 \downarrow & & \downarrow & \nearrow f' & \\
 B\mathbb{Z}/p^a & = & B\mathbb{Z}/p^a & &
 \end{array}$$

By Lemma 2.3, it is enough to show that f' is nullhomotopic. But again by Lemma 2.3 applied to the fibration on the left, we see that f' is nullhomotopic since \bar{f} restricted to $\langle x_k \rangle$ is so. Therefore \bar{f} is nullhomotopic.

Assume now that G is not finite. Any subgroup of G generated by a finite number of elements of order a power of p has a finite abelianization, and must therefore be finite itself by [20, Theorem 2.26]. Thus G is locally finite, i.e. G is a filtered colimit of finite nilpotent groups generated by elements of order p^i for $i \leq r$. Likewise, BG is a filtered homotopy colimit of BS where S are finite groups (generated by finite subsets of the set of generators) which verify the hypothesis of the proposition. The total space E can be obtained as a pointed filtered colimit of the total spaces obtained by pulling back the fibration. By the finite case situation they are all cellular and therefore so is E . \blacksquare

Sometimes the existence of the “local” sections defined for every generator permits to construct a global section of the fibration. By a result of Chachólski [7, Theorem 4.7] the total space of such a split fibration is cellular since F and BG are so. This is the case for an H -fibration and E is then weakly equivalent to a product $F \times BG$.

A straightforward consequence of the above proposition (in the case when the fibration is the identity of BG) is the following characterization of the $B\mathbb{Z}/p^r$ -cellular classifying spaces. For $r = 1$ we obtain R. Flores’ result [10].

Corollary 2.5. *Let $r \geq 1$ and G be a nilpotent group generated by elements of order p^i with $i \leq r$. Then BG is $B\mathbb{Z}/p^r$ -cellular.* \blacksquare

Example 2.6. The quaternion group Q_8 of order 8 is generated by elements of order 4. Therefore BQ_8 is $B\mathbb{Z}/4$ -cellular. We do not know an explicit way to construct BQ_8 as a pointed homotopy colimit of a diagram whose values are copies of $B\mathbb{Z}/4$.

The previous technical propositions allow us to state the main result of this section. It provides a constructive description of the cellularization of the total space of certain fibrations over classifying spaces of nilpotent groups.

Theorem 2.7. *Let G be a nilpotent group and $F \longrightarrow E \longrightarrow BG$ be a fibration with $B\mathbb{Z}/p^r$ -cellular fiber F . Then the cellularization of E is the total space of a fibration $F \longrightarrow CW_{B\mathbb{Z}/p^r}(E) \longrightarrow BS$ where $S \triangleleft G$ is the (normal) subgroup generated by the p -torsion elements g of order p^i with $i \leq r$ such that the inclusion $B\langle g \rangle \longrightarrow BG$ lifts to E up to unpointed homotopy.*

Proof. By Proposition 2.1 pulling back along $BS \rightarrow BG$ yields a cellular equivalence f in the following square:

$$\begin{array}{ccc} E_S & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ BS & \longrightarrow & BG \end{array}$$

By Proposition 2.4 the total space E_S is cellular and therefore $E_S \simeq CW_{B\mathbb{Z}/p^r}(E)$. ■

Corollary 2.8. *Let G be a nilpotent group and $S \triangleleft G$ be the (normal) subgroup generated by the p -torsion elements g of order p^i with $i \leq r$. Then $CW_{B\mathbb{Z}/p^r}BG \simeq BS$. Moreover when G is finitely generated, S is a finite p -group.*

Proof. We only need to show that S is a finite p -group. Notice that the abelianization of S is p -torsion, then S is also a torsion group (see [23, Cor. 3.13]). Moreover, since G is finitely generated, by [23, 3.10], S is finite. ■

In fact in that case the previous result also holds when the base space is an Eilenberg-Mac Lane space $K(G, n)$.

Proposition 2.9. *Let $F \xrightarrow{i} E \xrightarrow{\pi} K(G, n)$ be a fibration where G is a finitely generated group by elements of order p^i where $i \leq r$ and $n > 1$. Assume that for each generator $x \in G$, the inclusion $K(\langle x \rangle, n) \longrightarrow K(G, n)$ lifts to E . If F is $B\mathbb{Z}/p^r$ -cellular then E is so.* ■

3. CELLULARIZATION OF NILPOTENT POSTNIKOV PIECES

In this section we compute the cellularization with respect to $B\mathbb{Z}/p^r$ of nilpotent Postnikov pieces. The main difficulty lies in the fundamental group, so it will be no surprise that these results hold as well for cellularization with respect to $B^m\mathbb{Z}/p^r$ with $m \geq 2$. We will often use the following closure property [9, Theorem 2.D.11].

Proposition 3.1. *Let $F \rightarrow E \rightarrow B$ be a fibration where F and E are A -cellular. Then so is B .* ■

Example 3.2. [9, Corollary 3.C.10] The Eilenberg-Mac Lane space $K(\mathbb{Z}/p^k, n)$ is $B\mathbb{Z}/p^r$ -cellular for any integer k and any $n \geq 2$.

The construction of the cellularization is performed by looking first at the universal cover of the Postnikov piece. We start with the basic building blocs, the Eilenberg-Mac Lane spaces. For the results on the structure on infinite abelian groups, we refer the reader to Fuchs' book [12].

Lemma 3.3. *An Eilenberg-Mac Lane space $K(A, m)$ with $m \geq 2$ is $B\mathbb{Z}/p^r$ -cellular if and only if A is a p -torsion abelian group.*

Proof. That A must be p -torsion is clear. Assume thus that A is a p -torsion group. If A is bounded it is isomorphic to a direct sum of cyclic groups. Since cellularization commutes with finite products $K(A, m)$ is $B\mathbb{Z}/p^r$ -cellular when A is a finite direct sum of cyclic groups. Taking a (possibly transfinite) telescope of $B\mathbb{Z}/p^r$ -cellular spaces we obtain that $K(A, m)$ is so for any bounded group.

In general A splits as a direct sum of a divisible group D and a reduced one T . A p -torsion divisible group is a direct sum of copies of \mathbb{Z}/p^∞ , which is a union of bounded groups, thus $K(D, m)$ is cellular. Now T has a basic subgroup $P < T$ which is a direct sum of cyclic groups and the quotient T/P is divisible. So $K(T, m)$ is the total space of a fibration

$$K(P, m) \rightarrow K(T, m) \rightarrow K(D, m)$$

When $m \geq 3$ we are done because of the above mentioned closure property Proposition 3.1. If $m = 2$ we have to refine the analysis of the fibration because $K(D, m - 1)$ is not cellular. However, as D is a union of bounded groups $D[p^k]$, the space $K(T, 2)$ is the telescope of total spaces X_k of fibrations with cellular fiber $K(P, 2)$ and base $K(D[p^k], 2)$. We claim that these total spaces are cellular (and thus so is $K(T, 2)$) and proceed by induction on the bound. Consider the subgroup $D[p^k] < D[p^{k+1}]$ whose quotient is a direct sum of cyclic groups \mathbb{Z}/p . Therefore X_{k+1} sits in a fibration

$$K(\oplus \mathbb{Z}/p, 1) \rightarrow X_k \rightarrow X_{k+1}$$

where fiber and total space are cellular. We are done. ■

We are now ready to prove that any p -torsion simply connected Postnikov piece is a $B\mathbb{Z}/p^r$ -cellular space.

Proposition 3.4. *A simply connected Postnikov piece is $B\mathbb{Z}/p^r$ -cellular if and only if it is p -torsion.*

Proof. Let X be a simply connected p -torsion Postnikov piece. For some integer m , the m -connected cover $X\langle m \rangle$ is an Eilenberg-Mac Lane space, which is cellular by Lemma 3.3. Consider the principal fibration

$$K(\pi_m(X), m - 1) \longrightarrow X\langle m \rangle \longrightarrow X\langle m - 1 \rangle$$

If $m \geq 3$ both $X\langle m \rangle$ and $K(\pi_m(X), m - 1)$ are cellular. It follows that $X\langle m - 1 \rangle$ is cellular by the closure property Proposition 3.1. The same argument shows that $X\langle 2 \rangle$ is cellular.

Let us thus look at the fibration $X\langle 2 \rangle \rightarrow X \rightarrow K(\pi_2 X, 2)$. The same discussion on the p -torsion group $\pi_2 X$ as in Lemma 3.3 will apply. If this is a bounded group, say by p^k , an induction on the bound shows that X is actually the base space of a fibration where the total space is cellular because its second homotopy group is p^{k-1} -bounded, and the fiber is cellular because it is of the form $K(V, 1)$ with V a p -torsion abelian groups whose torsion is bounded by p^r . The closure property Proposition 3.1 ensures that X is then cellular.

If $\pi_2 X$ is divisible, X is a telescope of cellular spaces, hence cellular. If it is reduced, taking a basic subgroup $B < \pi_2 X$ yields a diagram of fibrations

$$\begin{array}{ccccc} X\langle 2 \rangle & \longrightarrow & Y & \longrightarrow & K(B, 2) \\ \downarrow & & \downarrow & & \downarrow \\ X\langle 2 \rangle & \longrightarrow & X & \longrightarrow & K(\pi_2 X, 2) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & K(D, 2) & = & K(D, 2) \end{array}$$

which exhibits X as the total space of a fibration over $K(D, 2)$ with D divisible and $B\mathbb{Z}/p^r$ -cellular fiber. Therefore writing D as a union of bounded groups as in the proof of Lemma 3.3, X is a telescope of cellular spaces, therefore it is $B\mathbb{Z}/p^r$ -cellular as well. \blacksquare

Remark 3.5. The proof of the proposition holds in the more general setting when X is a p -torsion space such that $X\langle m \rangle$ is $B\mathbb{Z}/p^r$ cellular for some $m \geq 2$. The proposition corresponds to the case when some m -connected cover $X\langle m \rangle$ is contractible.

Recall from [13, Corollary 2.12] that a connected space is nilpotent if and only if its Postnikov system admits a principal refinement $\cdots \rightarrow X_s \rightarrow X_{s-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0$. This means that each map $X_{s+1} \rightarrow X_s$ in the tower is a principal fibration with fiber $K(A_s, i_s - 1)$ for some increasing sequence of integers $i_s \geq 2$. We are only interested in finite Postnikov pieces, i.e. nilpotent spaces that can be constructed in a finite number of steps by taking homotopy fibers of k -invariants $X_s \rightarrow K(A_s, i_s)$.

The key step in the study of the cellularization of a nilpotent finite Postnikov piece is the analysis of a principal fibration (given in our case by the k -invariants).

Theorem 3.6. *Let X be a p -torsion nilpotent Postnikov piece. Then there exists a fibration*

$$X\langle 1 \rangle \longrightarrow CW_{B\mathbb{Z}/p^r} X \longrightarrow BS$$

where S is the (normal) subgroup of $\pi_1(X)$ generated by the elements g of order p^i with $i \leq r$ such that the inclusion $B\langle g \rangle \rightarrow B\pi_1 X$ admits a lift to X up to unpointed homotopy.

Proof. By Proposition 3.4 the universal cover $X\langle 1 \rangle$ is cellular and there is a fibration $X\langle 1 \rangle \longrightarrow X \longrightarrow BG$ where $G = \pi_1(X)$ is nilpotent. The result follows then from Theorem 2.7. \blacksquare

4. CELLULARIZATION OF H -SPACES

In this section we will apply the computations of the cellularization of p -torsion nilpotent Postnikov systems to determine $CW_{B\mathbb{Z}/p}X$ when X is an H -space. We prove:

Theorem 4.1. *Let X be a connected H -space such that $\Omega^n X$ is $B\mathbb{Z}/p$ -local. Then*

$$CW_{B\mathbb{Z}/p}X \simeq Y \times K(W, 1)$$

where Y is a simply connected p -torsion H -Postnikov piece with homotopy concentrated in degrees $\leq n$ and W is an elementary abelian p -group.

Proof. The fibration in Bousfield's result Proposition 1.5 yields a cellular equivalence between a connected p -torsion H -Postnikov piece F and X . Theorem 3.6 thus applies. Moreover, as F is an H -space as well, the subgroup S is abelian generated by elements of order p . Therefore we have a fibration $F\langle 1 \rangle \rightarrow CW_{B\mathbb{Z}/p}F \rightarrow K(W, 1)$ which admits a section (summing up the local section). The cellularization therefore splits. \blacksquare

This result applies for H -spaces satisfying certain finiteness conditions.

Proposition 4.2. *Let X be a connected H -space such that $H^*(X; \mathbb{F}_p)$ is finitely generated as algebra over the Steenrod algebra. Then*

$$CW_{B\mathbb{Z}/p}X \simeq F \times K(W, 1)$$

*where F is a 1-connected p -torsion H -Postnikov piece and W is an elementary abelian p -group. Moreover there exists an integer k such that $CW_{B^m\mathbb{Z}/p}X \simeq *$ for $m \geq k$.*

Proof. In [6] the authors prove that if $H^*(X; \mathbb{F}_p)$ is finitely generated as algebra over the Steenrod algebra then $\Omega^n X$ is $B\mathbb{Z}/p$ -local for some $n \geq 0$. Hence Theorem 4.1 applies and we obtain the desired result. In addition Lemma 1.1 shows that X is $B^{n+s+1}\mathbb{Z}/p$ -local for any $s \geq 0$, which implies the second part of the result. \blacksquare

The technique we propose in this paper is not only a nice theoretical tool which provides a general statement about how the $B\mathbb{Z}/p$ -cellularization of H -spaces look like. Our next result shows that one can actually identify precisely this new space when dealing with connected covers of finite H -spaces. Recall that by Miller's theorem [17, Thm. A], any finite H -space X is $B\mathbb{Z}/p$ -local and therefore $CW_{B\mathbb{Z}/p}(X) \simeq *$. The universal cover of X is still finite and thus $CW_{B\mathbb{Z}/p}(X\langle 1 \rangle)$ is contractible as well. We can therefore assume that X is 1-connected. The computation of the cellularization of the 3-connected cover is already implicit in [4].

Proposition 4.3. *Let X be a simply connected finite H -space and let k denote the rank of the free abelian group $\pi_3 X$. Then $CW_{B\mathbb{Z}/p}X\langle 3 \rangle \simeq K(\oplus_k \mathbb{Z}/p, 1)$. For $n \geq 4$, up to p -completion, the universal cover of $CW_{B\mathbb{Z}/p}(X\langle n \rangle)$ is weakly equivalent to the 2-connected cover of $\Omega(X[n])$.*

Proof. By Browder's famous result [5, Theorem 6.11] X is even 2-connected and its third homotopy group $\pi_3 X$ is free abelian (of rank k) by J. Hubbuck and R. Kane's theorem [14]. This means we have a fibration

$$K(\oplus_k \mathbb{Z}_{p^\infty}, 1) \longrightarrow X\langle 3 \rangle \longrightarrow P_{B\mathbb{Z}/p}X\langle 3 \rangle$$

which shows that $CW_{B\mathbb{Z}/p}X\langle 3 \rangle \simeq K(\oplus_k \mathbb{Z}/p, 1)$. We deal now with the higher connected covers. Consider the following commutative diagram of fibrations

$$\begin{array}{ccccc}
 F & \longrightarrow & F & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega X[n] & \longrightarrow & X\langle n \rangle & \longrightarrow & X \\
 \downarrow & & \downarrow & & \parallel \\
 P_{B\mathbb{Z}/p}(\Omega X[n]) & \longrightarrow & P_{B\mathbb{Z}/p}(X\langle n \rangle) & \longrightarrow & X
 \end{array}$$

where F is a p -torsion Postnikov piece by [2, Thm 7.2] and the fiber inclusions are all $B\mathbb{Z}/p$ -cellular equivalences because the base spaces are $B\mathbb{Z}/p$ -local. Therefore

$$CW_{B\mathbb{Z}/p}(X\langle n \rangle) \simeq CW_{B\mathbb{Z}/p}(F) \simeq F\langle 1 \rangle \times K(W, 1)$$

We wish to identify $F\langle 1 \rangle$. Since the fibrations in the diagram are nilpotent, by [3, II.4.8] they remain fibrations after p -completion. By Neisendorfer's theorem [19] the map $P_{B\mathbb{Z}/p}(X\langle n \rangle) \rightarrow X$ is an equivalence up to p -completion, which means that $P_{B\mathbb{Z}/p}(\Omega(X[n]))_p^\wedge \simeq *$. Thus $F_p^\wedge \simeq (\Omega(X[n]))_p^\wedge$. Notice that $\Omega(X[n])$ is 1-connected and its second homotopy group is free by the above mentioned theorem of Hubbuck and Kane (which corresponds up to p -completion to the direct sum of k copies of the Prüfer group \mathbb{Z}/p^∞ in $\pi_1 F$). Hence $F\langle 1 \rangle$ coincides with $\Omega(X[n])\langle 2 \rangle$ up to p -completion. \blacksquare

To illustrate this result we compute the $B\mathbb{Z}/2$ -cellularization of the successive connected covers of S^3 . The only delicate point is the identification of the fundamental group.

Example 4.4. Recall that S^3 is $B\mathbb{Z}/2$ -local since it is a finite space. Thus the cellularization $CW_{B\mathbb{Z}/2}S^3$ is contractible. Next the fibration

$$K(\mathbb{Z}_{2^\infty}, 1) \rightarrow S^3\langle 3 \rangle \rightarrow P_{B\mathbb{Z}/2}S^3\langle 3 \rangle$$

shows that $CW_{B\mathbb{Z}/2}S^3\langle 3 \rangle \simeq K(\mathbb{Z}/2, 1)$. Finally since $S^3[4]$ does not split as a product (the k -invariant is not trivial), we see that $CW_{B\mathbb{Z}/2}S^3\langle 4 \rangle \simeq K(\mathbb{Z}/2, 3)$. Likewise, for any integer $n \geq 4$, we have that $CW_{B\mathbb{Z}/2}S^3\langle n \rangle$ is equivalent to the 2-completion of the 2-connected cover of $\Omega(S^3[n])$. The same phenomenon occurs at odd primes.

5. CELLULARIZATION WITH RESPECT TO $B^m\mathbb{Z}/p$

All the techniques developed for fibrations over BG apply to fibrations over $K(G, n)$ when $n > 1$ and we get the following results.

Lemma 5.1. *Let $n \geq 2$ and X be a connected space. Then*

$$CW_{B^n\mathbb{Z}/p^r}(X) = CW_{B^n\mathbb{Z}/p^r}(X\langle n-1 \rangle).$$

Proof. For $n > i$ consider the following fibrations over $B^n\mathbb{Z}/p^r$ -local base spaces:

$$X\langle i \rangle \longrightarrow X\langle i-1 \rangle \longrightarrow K(\pi_i(X), i)$$

We see that $CW_{B^n\mathbb{Z}/p^r}(X\langle i \rangle) = CW_{B^n\mathbb{Z}/p^r}(X\langle i-1 \rangle)$. ■

Proposition 5.2. *Let $m \geq 2$ and X be a p -torsion nilpotent Postnikov piece. Then there exists a fibration*

$$X\langle m \rangle \longrightarrow CW_{B^m\mathbb{Z}/p}X \longrightarrow K(W, m)$$

where W is a p -torsion subgroup of $\pi_m(X)$ with torsion bounded by p^r .

Theorem 5.3. *Let X be a connected H -space such that $\Omega^n X$ is $B^m\mathbb{Z}/p$ -local. Then*

$$CW_{B^m\mathbb{Z}/p}X \simeq F \times K(W, m)$$

where F is a p -torsion H -Postnikov piece with homotopy concentrated in degrees from $m+1$ to n and W is an elementary abelian p -group.

Example 5.4. Let X denote “Milgram’s space”, see [16], the fiber of $Sq^2 : K(\mathbb{Z}/2, 2) \rightarrow K(\mathbb{Z}/2, 4)$. This is an infinite loop space. By Proposition 3.4 we know it is already $B\mathbb{Z}/2$ -cellular. Let us compute the cellularization with respect to $B^m\mathbb{Z}/2$ for higher m ’s. Since the k -invariant is not trivial, we see that $CW_{B^2\mathbb{Z}/2}X \simeq CW_{B^3\mathbb{Z}/2}X \simeq K(\mathbb{Z}/2, 3)$.

We compute finally the cellularization of the (infinite loop) space BU and its 2-connected cover BSU with respect to Eilenberg-MacLane spaces $B^m\mathbb{Z}/p$. By Bott periodicity this actually tells us the answer for all connected covers of BU .

Example 5.5. First of all, recall from Example 1.4 that BU is $B^2\mathbb{Z}/p$ -local since $\tilde{K}^*(B^2\mathbb{Z}/p) = 0$ and its iterated loops are never $B\mathbb{Z}/p$ -local. Therefore $CW_{B^m\mathbb{Z}/p}(BU)$ is contractible if $m \geq 2$. Since $BU \simeq BSU \times BS^1$ the same property holds for BSU .

We now compute the $B^m\mathbb{Z}/p$ -cellularization of BO and its connected covers BSO , $BSpin$, and $BString$.

Proposition 5.6. *Let $m \geq 2$. Then*

- (i) $CW_{B^m\mathbb{Z}/p}(BO) \simeq CW_{B^m\mathbb{Z}/p}(BSO) \simeq CW_{B^m\mathbb{Z}/p}(BSpin) \simeq *$,
- (ii) $CW_{B^m\mathbb{Z}/p}(BString) \simeq *$ if $m > 2$,
- (iii) $CW_{B^2\mathbb{Z}/p}(BString) \simeq K(\mathbb{Z}/p, 2)$ and $\text{map}_*(B^2\mathbb{Z}/p, BString) \simeq \mathbb{Z}/p$.

Proof. In [15] W. Meier proves that real and complex K -theory have the same acyclic spaces, hence BO is also $B^2\mathbb{Z}/p$ -local. Therefore $CW_{B^m\mathbb{Z}/p}(BO)$ is contractible for any $m \geq 2$. The 2-connected cover of BO is BSO and there is a splitting $BO \simeq BSO \times B\mathbb{Z}/2$, so that $CW_{B^m\mathbb{Z}/p}(BSO) \simeq *$.

The 4-connected cover of BO is $BSpin$. It follows from the fibration

$$BSpin \rightarrow BSO \xrightarrow{w_2} K(\mathbb{Z}/2, 2)$$

that the fiber of $BSpin \rightarrow BSO$ is $B\mathbb{Z}/2$. Since BSO and $B\mathbb{Z}/2$ are $B^2\mathbb{Z}/p$ -local, so is $BSpin$. Therefore $CW_{B^m\mathbb{Z}/p}(BSpin)$ is contractible.

Finally, the 8-connected cover of BO is $BString$. It is the homotopy fiber of $BSpin \xrightarrow{p_1/4} K(\mathbb{Z}, 4)$, where p_1 denotes the first Pontrjagin class. Consider the fibration

$$K(\mathbb{Z}, 3) \rightarrow BString \rightarrow BSpin$$

where the base space is $B^m\mathbb{Z}/p$ -local for $m \geq 2$. Together with the exact sequence $\mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}] \rightarrow \mathbb{Z}/p^\infty$, this implies that

$$CW_{B^m\mathbb{Z}/p}(BString) \simeq CW_{B^m\mathbb{Z}/p}(K(\mathbb{Z}, 3)) \simeq CW_{B^m\mathbb{Z}/p}(K(\mathbb{Z}/p^\infty, 2))$$

which is contractible unless $m = 2$, when we obtain $K(\mathbb{Z}/p, 2)$. This yields the explicit description of the pointed mapping space $\text{map}_*(B^2\mathbb{Z}/p, BString)$. \blacksquare

Observe that the iterated loops of the m -connected covers of BO and BU are never $B\mathbb{Z}/p$ -local. Hence we know that their cellularization with respect to $B\mathbb{Z}/p$ must have infinitely many non-vanishing homotopy groups by Proposition 1.6.

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